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Tests for Independence of Two Multivariate Regression Equations with Different Design Matrices*

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In this paper, the authors considered various procedures for testing for the independence of two multivariate regression equations with different design matrices. Asymptotic null distributions as well as nonnull distributions under local alternatives of the test statistics associated with the above procedures are also derived.

1. INTRODUCTION

Considerable amount of work was done in the literature on the problems of estimation and testing of hypotheses under the classical multivariate regression model. But this work is primarily done under the assumption that the design matrix remains the same for all variables. In this paper, we derive asymptotic distributions of test statistics for independence of two multivariate regression equations with different design matrices. But the assumption of the same design matrix is unrealistic in some situations since

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the nature of some variables is different from others. For example, we may represent consumption investment models of two countries by regression equations with different design matrices but the above equations may be correlated due to the trade between the two countries. To give another example, the performance of complicated systems on certain variables like speed, accuracy, etc., may be predicted by studying the performance of their components by using regression equations with different design matrices. The dependent variables here may be correlated. Correlated regression equations with different design matrices arise when some of the observations on certain variables are missing. Some work was done (e.g., Srivastava [9] and Trawinski [12]) on this problem but we do not deal with missing data in this paper. Very little work was done in the literature in the area of inference on multivariate regression equations with different design matrices for different sets of variables. When we have correlated univariate regression equations with different design matrices, the above equations are known in the econometric literature as seemingly unrelated regression equations. Zellner [14, 15] and some other econometricians worked on the estimation problems in this situation. Several authors have investigated the efficiency of the generalized least square (GLS) estimates of the regression coefficients of two correlated regression equations by replacing the unknown covariance matrix with different sample estimates. It was found (e.g., see Kmenta and Gilbert [5], Revankar [7]) that the GLS estimates are more efficient than the ordinary least square (OLS) estimates except in the situations when the sample size is small and/or the error vectors of the two equations are nearly uncorrelated. A review of the work done by econometricians was given in Srivastava and Dwivedi [10]. Kariya [4] derived a locally best invariant (LBI) test and a locally best invariant unbiased (LBIU) test for the hypothesis of independence of the two univariate regression equations. But very little work was done on the problems of inference of correlated multivariate regression equations.

In Section 3 of the paper, we propose a locally best invariant (LBI) test for the independence of two multivariate regression equations with different design matrices. A derivation of this test is given in Section 4. In Section 5, we derive asymptotic null distributions of the LBI test statistic and two other test statistics useful for testing the independence of two multivariate regression equations. The asymptotic nonnull distributions of the test statistics considered in Section 5 are derived in Section 6 under local alternatives. The expressions derived in Sections 4 and 5 are linear combinations of chi-square variables. The discussions in Sections 3–6 are restricted to the tests against 2-sided alternatives. In Section 7, we discuss the procedures for testing for the independence of two multivariate regression equations against 1-sided alternatives.

2. Preliminaries and Statement of Problems

Consider the following correlated multivariate regression equations

$$Y_i = X_i \theta_{ii} + E_i \qquad (i = 1, 2, \ldots, q), \tag{2.1}$$

where the design matrices X_i: $n \times r_i$ are known and the matrices $\theta_i : r_i \times p_i$ of the parameters are unknown for $i = 1, 2, \ldots, q$. Also, let $E_i = (\mathbf{e}_{i1}, \ldots, \mathbf{e}_{ip_i})$, where $\mathbf{e}_{ij}$ is of order $n \times 1$. We assume that $\mathbf{e}'$: $1 \times pn$ is distributed as a multivariate normal with mean vector $\mathbf{0}$ and covariance matrix $\Sigma_0 = \Sigma \otimes I_n$, where $\otimes$ denotes the Kronecker product,

$$\mathbf{e}' = (\mathbf{e}'_{11}, \ldots, \mathbf{e}'_{1p_1}, \ldots, \mathbf{e}'_{q1}, \ldots, \mathbf{e}'_{qp_q}),$$

$\Sigma = (\Sigma_{ij})$: $p \times p$, $p = p_1 + \cdots + p_q$ and Σ_{ij} is of order $p_i \times p_j$. The usual MANOVA model is given by

$$Y = X_0 \theta_0 + E, \tag{2.2}$$

where $Y = [Y_1, \ldots, Y_q]$, $\theta_0 = [\theta_{11}, \ldots, \theta_{qq}]$, and $E = [E_1, \ldots, E_q]$. In the above model, the design matrix X_0: $n \times r$ is the same for each of the q sets of variables. But, in the model (2.1), the design matrices $X_1, \ldots, X_q$ are different for different sets of variables even if $r_1 = \cdots = r_q = r$. In this sense, the model given by (2.1) is more general than the model (2.2). The model with correlated growth curve equations (see Krishnaiah [6]) is more general than the model given by (2.1).

Next, consider the model

$$Y = X\theta + E, \tag{2.3}$$

where $X = [X_1, \ldots, X_q]$, $\theta = [\theta_{ij}]$, and θ_{ij} is of order $r_i \times p_j$. The model (2.3) is the usual MANOVA model when θ is completely unknown. But, when some of the elements of θ are known, we have to take advantage of this knowledge and this poses a different set of problems. In particular, when $\theta_{ij} = 0$ for $i \neq j$, the model (2.3) is the same as the model (2.1). The model (2.1) may be referred to as the MANOVA model with diffeent design matrices or MANOVA model with partial knowledge of location parameters. It basically involves correlated multivariate regression equations with different design matrices and may be also referred to as the CMRE model.

We can express the model (2.1) in the following way when $p_1 = \cdots = p_q = p_0$,

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 0 & 0 & \cdots & X_q \end{pmatrix} \begin{pmatrix} \theta_{11} \\ \theta_{22} \\ \vdots \\ \theta_{qq} \end{pmatrix} + \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{pmatrix}. \tag{2.4}$$

In the usual MANOVA model, all the columns of $[E_1',..., E_q']$ are distributed independently. But, in our case, this is not true. So we cannot treat it as the usual MANOVA model. The model (2.1) can be written as follows whether p_i's are equal or not,

$$\mathbf{y} = \tilde{X}\boldsymbol{\theta} + \mathbf{e}, \tag{2.5}$$

where $\mathbf{y}' = (\mathbf{y}_1',..., \mathbf{y}_q')$, $\mathbf{y}_i' = (\mathbf{y}_{i1}',..., \mathbf{y}_{ip_i}')$, and $\mathbf{y}_{ij}$ is jth column of Y_i. Also, $\boldsymbol{\theta}' = (\boldsymbol{\theta}_1',..., \boldsymbol{\theta}_q')$, $\boldsymbol{\theta}_j' = (\boldsymbol{\theta}_{jj1}',..., \boldsymbol{\theta}_{jjp_j}')$, $\boldsymbol{\theta}_{jjt}$ is tth column of θ_{jj}, and

$$\tilde{X} = \mathrm{diag}(I_{p_1} \otimes X_1,..., I_{p_q} \otimes X_q), \tag{2.6}$$

where $\mathrm{diag}(A_1,..., A_q)$ is the block diagonal matrix with A_i's as diagonal blocks.

It is well known that the ordinary least square (OLS) estimate of θ_{ii}'s under the model (2.1) are given by

$$\hat{\theta}_{ii} = (X_i' X_i)^{-1} X_i' Y_i. \tag{2.7}$$

When Σ is known, the generalized least square (GLS) estimate of $\boldsymbol{\theta}$ is given by

$$(\tilde{X}'(\Sigma^{-1} \otimes I_n)\tilde{X})^{-1} \tilde{X}'(\Sigma^{-1} \otimes I_n)\mathbf{y}. \tag{2.8}$$

When Σ is unknown, a modified GLS estimate of $\boldsymbol{\theta}$ is given by

$$\hat{\hat{\boldsymbol{\theta}}} = (\tilde{X}'(\hat{\Sigma}^{-1} \otimes I_n)\tilde{X})^{-1} \tilde{X}'(\hat{\Sigma}^{-1} \otimes I_n)\mathbf{y}, \tag{2.9}$$

where $\hat{\Sigma}$ is a suitable estimate of Σ. When $p_1 = \cdots = p_q = 1$, several authors have investigated the efficiency of $\hat{\hat{\boldsymbol{\theta}}}$ by using different choices of $\hat{\Sigma}$. One possible estimate of $\hat{\Sigma}$ is given by

$$\hat{\Sigma} = \frac{1}{n}\begin{pmatrix} \hat{E}_1' \\ \hat{E}_2' \\ \vdots \\ \hat{E}_q' \end{pmatrix} [\hat{E}_1, \hat{E}_2 \cdots \hat{E}_q] = \frac{1}{n}\begin{bmatrix} Y_1'Q_1Y_1 & Y_1'Q_1Q_2Y_2 & \cdots & Y_1'Q_1Q_qY_q \\ Y_2'Q_2Q_1Y_1 & Y_2'Q_2Y_2 & \cdots & Y_2'Q_2Q_qY_q \\ \vdots & \vdots & & \vdots \\ Y_q'Q_qQ_1Y_1 & Y_q'Q_qQ_2Y_2 & \cdots & Y_q'Q_qY_q \end{bmatrix}, \tag{2.10}$$

where

$$\hat{E}_i = Y_i - X_i\hat{\theta}_{ii} = Q_iY_i \tag{2.11}$$

and

$$Q_i = I_n - P_i \qquad \text{and} \qquad P_i = X_i(X_i'X_i)^{-1}X_i'. \tag{2.12}$$

In this paper, we consider the problem of testing the hypothesis $\Sigma_{12}=0$ under the model (2, 1) with $q=2$ and

$$\Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}, \tag{2.13}$$

where Σ_{12} is of order $p_1 \times p_2$. The problem of testing the hypothesis that $\Sigma_{12}=0$, and $\theta_{ij}=0$ $(i \neq j)$ under the model (2.3) will be discussed by the authors in a subsequent publication.

3. Locally Best Invariant Test for Independence of Two Regression Equations

In this section, we reduce via invariance, the problem of testing the independence hypothesis $\mathscr{H}: \Sigma_{12}=0$ of two multivariate regression equations $Y_i=X_i\theta_{ii}+E_i$ (i = 1, 2) to canonical form and derive the LBI test. Further, based on the form of the LBI test, we also propose some other tests.

As is easily observed, the problem is left invariant under group $\tilde{G}=\mathscr{O}(n) \times Gl(p_1) \times Gl(p_2) \times R^{r_1p_1} \times R^{r_2p_2}$ acting on the model by

$$Y_i \to CY_iB_i+X_iF_i \qquad (i=1,2), \tag{3.1}$$

where $(C, B_1, B_2, F_1, F_2) \in \tilde{G}$, and $\tilde{G}$ acts on the parameter space $(\theta_{11}, \theta_{22}, \Sigma)$ by

$$\theta_{ii} \to \theta_{ii}B_i+F_i \quad \text{and} \quad \Sigma_{ij} \to B_i'\Sigma_{ij}B_j \; (i,j=1,2). \tag{3.2}$$

Here $\mathscr{O}(n)$ and $Gl(p)$ denote, respectively, the set of $n \times n$ orthogonal matrices and the set of $p \times p$ nonsingular matrices. Then it is shown that a maximal invariant is

$$(Z_1'Y_1(Y_1'Q_1Y_1)^{-1/2}, Z_2'Y_2(Y_2'Q_2Y_2)^{-1/2}), \tag{3.3}$$

where $(Y_i'Q_iY_i)^{-1/2} \in \mathscr{S}(p_i)$, $Q_i=I_n-P_i=I_n-X_i(X_i'X_i)^{-1}X_i'$, and Z_i is an $n \times (n-r_i)$ fixed matrix satisfying

$$Q_i=Z_iZ_i' \qquad \text{and} \qquad Z_i'Z_i=I_{n-r_i}\,(i=1,2). \tag{3.4}$$

Here $\mathscr{S}(p)$ denotes the set of $p \times p$ positive definite matrices. From (3.3), any maximal invariant is a function of

$$W_i=Z_i'Y_i \qquad (i=1,2). \tag{3.5}$$

On the other hand, a maximal invariant parameter is the vector of canonical correlations ρ_i's, where the ρ_i^2's are the latent roots of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Without loss of generality, we assume that $p_1 \geqslant p_2$ and let $D_\rho = \operatorname{diag}(\rho_1,..., \rho_{p_2})$ and

$$\Lambda = \begin{bmatrix} D_\rho \\ 0 \end{bmatrix} : p_1 \times p_2. \tag{3.6}$$

Then the distribution of a maximal invariant depends on $(\theta_{11}, \theta_{22}, \Sigma)$ only through Λ, and hence we may assume, without loss of generality,

$$\Sigma = \begin{pmatrix} I & \Lambda \\ \Lambda' & I \end{pmatrix}$$

when an invariant test is treated. We use the following notation in the sequel

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} W_1'W_1 & W_1'Z_1'Z_2W_2 \\ W_2'Z_2'Z_1W_1 & W_2'W_2 \end{pmatrix} \tag{3.7}$$

and

$$\tau = \operatorname{tr} \Lambda\Lambda' = \Sigma_{i=1}^{p_2} \rho_i^2.$$

The main theorem of this section is

THEOREM 3.1. *For any invariant test ϕ of level α, the power function is evaluated as*

$$E_\Lambda[\phi] = \alpha + \frac{\tau}{2} E_0[J\phi] + o(\phi, \Lambda), \tag{3.8}$$

where

$$J = \frac{n_1 n_2}{p_1 p_2} \operatorname{tr} S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} - \frac{n_1}{p_1} \operatorname{tr} S_{11}^{-1} W_1' Z_1' Z_2 Z_2' Z_1 W_1$$

$$- \frac{n_2}{p_2} \operatorname{tr} S_{22}^{-1} W_2' Z_2' Z_1 Z_1' Z_2 W_2, \tag{3.9}$$

$$\lim_{\tau \to 0} \sup_{\phi} o(\phi, \Lambda)/\tau = 0 \qquad \text{and} \qquad n_i = n - r_i.$$

The α level test which rejects for $J > k$ is a LBI level α test.

The proof of the above theorem is given in the next section. When $p_1 = p_2 = 1$, the test based on the statistic in (3.9) is the locally best invariant unbiased (LBIU) test derived by Kariya [4]. But when $p_1 = p_2 = 1$, our group $\tilde{G} = R \times R \times R^{r_1} \times R^{r_2}$ is larger than the group $R_+ \times R_+ \times R^{r_1} \times R^{r_2}$ taken in that paper, and so the test based on J in (3.9) is simply LBI.

In (3.9), the first term, except for a constant, is the trace of the canonical correlation matrix based on the residual $\hat{E}_i = Q_i Y_i$, which has intuitive appeal. But the second and third terms are hard to interpret. Of course, if $Z_1' Z_2 Z_2' Z_1 = I_{p_1}$ and $Z_2' Z_1 Z_1' Z_2 = I_{p_2}$ or, equivalently, if $X_1(X_1'X_1)^{-1} X_1' = X_2(X_2'X_2)^{-1} X_2'$, both of the terms become constants. It is noted that $X_1(X_1'X_1)^{-1} X_1' = X_2(X_2'X_2)^{-1} X_2'$ is a necessary and sufficient condition for a GLSE to be identically equal to each of the LSE's (see Kariya [4]). On the other hand, even if X_1 is a subset of X_2, the first term is not reduced to a constant. A comparison of this fact with the LBI test in the MANOVA model implies that when the information on the regression coefficients is available, the LBI test uses the information. In fact, in the MANOVA model where $X_1 = X_2$, $\operatorname{tr} S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}$ is the LBI test (see Schwartz [8]) and in addition, if $p_1 = p_2 = 1$, it is UMPI (uniformly most powerful invariant) test. In our problem, even if $p_1 = p_2 = 1$, there exists no UMPI test.

Next, analogous to the tests of independence in the MANOVA model, even in our problem, we may propose tests with the following critical regions:

(i) trace test: $\operatorname{tr} S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} > c$

(ii) LRT-like-test: $|I - S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}| < c$.

The distributions of these test statistics as well as the LBI test statistic in (3.9) will be considered in Sections 5 and 6.

Finally it is pointed out that the distribution of each of the second and third terms of (3.9) is marginally parameter-free though they are a part of a sufficient statistic. This implies that they are ancillary statistics obtained through the information on the regression coefficients and the effect of having the information on the LBI test appears in these terms compared to the LBI test in the MANOVA model. In the MANOVA model, no ancillary statistic exists.

4. Derivation of LBI Test: Proof of Theorem 3.1

A maximal invariant is a function of $W_i = Z_i' Y_i$ $(i = 1, 2)$ and on W_i's, $G = Gl(p_1) \times Gl(p_2)$ acts by $(W_1, W_2) \to (W_1 B_1, W_2 B_2)$. To derive the distribution of a maximal invariant, we apply the following result of Wijsman [13] on the probability ratio of the distributions of a maximal invariant.

Lemma 4.1. *Let $T = T(W_1, W_2)$ be any maximal invariant and let P_Λ^T be the distribution of T under Λ. Then the probability ratio of P_Λ^T and P_0^T evaluated at $T = T(W_1, W_2)$ is given by $(dP_\Lambda^T / dP_0^T)(T(W_1, W_2)) = N_\Lambda / N_0$,* where

$$N_\Lambda = \int_G f(g(W_1, W_2)|\Lambda)\,\chi(g)\,\nu(dg), \tag{4.1}$$

f is the density of (W_1, W_2), $g = (B_1, B_2) \in G$, and with $n_i = n - r_i$,

$$\chi(g) = \prod_{i=1}^{2} |B_i' B_i|^{n_i/2}, \qquad \nu(dg) = \prod_{i=1}^{2} |B_i' B_i|^{-p_i/2}\, dB_i. \tag{4.2}$$

Wijsman [13] states the condition for which Lemma (4.1) holds. However, it is easily checked in a similar way as in Kariya [3].

To evaluate the ratio N_Λ/N_0, let $\mathbf{w}_{ij}'$ be the jth row of W_i ($j = 1,..., n_i$: $i = 1, 2$), $\mathbf{w}^{(i)} = (\mathbf{w}_{i1}',..., \mathbf{w}_{in_i}')'$: $n_i p_i \times 1$ ($i = 1, 2$), and $\mathbf{w} = (\mathbf{w}^{(1)'}, \mathbf{w}^{(2)'})'$: $(n_1 p_1 + n_2 p_2) \times 1$. Then from $E = [E_1, E_2] \sim N(\mathbf{0}, I_n \otimes \Sigma)$, $\mathbf{w} \sim N(\mathbf{0}, A)$ with

$$A = \begin{pmatrix} I_{p_1} \otimes I_{p_1} & Z_1' Z_2 \otimes \Lambda \\ Z_2' Z_1 \otimes \Lambda' & I_{p_2} \otimes I_{p_2} \end{pmatrix}, \tag{4.3}$$

where $E \sim N\ (\mathbf{0}, I_n \otimes \Sigma)$ denotes that the row vector formed by placing the row vectors of E one after another is distributed as multivariate normal with mean vector $\mathbf{0}'$ and covariance matrix $I_n \otimes \Sigma$. Also, G acts on $\mathbf{w}$ by

$$g\mathbf{w} = \begin{pmatrix} I_{n_1} \otimes B_1' & 0 \\ 0 & I_{n_2} \otimes B_2' \end{pmatrix} \mathbf{w} \tag{4.4}$$

with $g = (B_1, B_2)$. Writing A^{-1} as $A^{-1} = [C^{ij}]$ with C^{ij}: $n_i p_i \times n_j p_j$ ($i, j = 1, 2$), we have from (4.2) and (4.3),

$$\begin{aligned} N_\Lambda = c\,|A|^{-1/2} \int_G \exp[-\tfrac{1}{2}\Sigma_{i,j=1}^2 \mathbf{w}^{(i)\prime} (I_{n_i} \otimes B_i) \\ \times C^{ij} (I_{n_j} \otimes B_j') \mathbf{w}^{(j)}]\,\chi(g)\,\nu(dg). \end{aligned} \tag{4.5}$$

Here using the identity $(I - A)^{-1} = I + A(I - A)^{-1} = I + A + A^2(I - A)^{-1}$,

$$\begin{aligned} C^{ii} &= I_{n_i} \otimes I_{p_i} + [H_i \otimes \Delta_i] + [H_i^2 \otimes \Delta_i^2]\, C^{ii} \qquad (i = 1, 2), \\ C^{12} &= -(Z_1' Z_2 \otimes \Lambda) - [Z_1' Z_2 H_2 \otimes \Lambda \Delta_2]\, C^{22}, \end{aligned} \tag{4.6}$$

where $C^{ii} = (I - H_i \otimes \Delta_i)^{-1}$ and $C^{12} = -(Z_1' Z_2 \otimes \Lambda)\, C^{22}$ are used, and

$$H_1 = Z_1' Z_2 Z_2' Z_1, \qquad H_2 = Z_2' Z_1 Z_1' Z_2, \qquad \Delta_1 = \Lambda \Lambda', \qquad \text{and} \qquad \Delta_2 = \Lambda' \Lambda. \tag{4.7}$$

Since $S_{ij} = W_i' Z_i' Z_j W_j$ (see (3.7)), from (4.6) we can write

$$\mathbf{w}^{(i)\prime}[I_{p_i} \otimes B_i]\, C^{ii}[I_{p_i} \otimes B_i']\, \mathbf{w}^{(i)} = \operatorname{tr} B_i' S_{ii} B_i + \operatorname{tr} B_i' W_i' H_i W_i \Delta_i + f_{ii}(B_i) \quad (i = 1, 2), \tag{4.8}$$

$$\mathbf{w}^{(1)\prime}[I_{p_1} \otimes B_1]\, C^{12}[I_{p_i} \otimes B_2']\, \mathbf{w}^{(2)} = -\operatorname{tr} B_1' S_{12} B_2 \Lambda + f_{12}(B_1, B_2),$$

where

$$\begin{aligned} f_{ii} &\equiv f_{ii}(B_i) = \mathbf{w}^{(i)\prime}[I_{n_i} \otimes B_i]\, [H_i^2 \otimes \Delta_i^2]\, C^{ii}[I \otimes B_i']\, \mathbf{w}^{(i)} \quad (i = 1, 2), \\ f_{12} &\equiv f_{12}(B_1, B_2) = \mathbf{w}^{(1)\prime}[I_{n_1} \otimes B_1]\, [Z_1' Z_2 H_2 \otimes \Lambda \Delta_2]\, C^{22}[I_{n_2} \otimes B_2']\, \mathbf{w}^{(2)}. \end{aligned} \tag{4.9}$$

Substituting these into (4.5) and replacing B_i by $S_{ii}^{-1/2} B_i$ with $S_{ii}^{-1/2} \in \mathscr{S}(p_i)$ yields

$$N_\Lambda = c \left(\prod_{i=1}^{2} |S_{ii}|^{-n_i/2} \right) |\Lambda|^{-1/2} \int_G \exp\left[-\tfrac{1}{2} \sum_{i=1}^{2} \operatorname{tr} B_i' B_i \right] \times \exp[K]\, \chi(g)\, \nu(dg), \tag{4.10}$$

where

$$K = \sum_{i=1}^{2} (K_{ii} + f_{ii}^*) + K_{12} + f_{12}^*, \qquad f_{ii}^* = f_{1i}(S_{ii}^{-1/2} B_i) \qquad (i = 1, 2),$$

$$f_{12}^* = f_{12}(S_{11}^{-1/2} B_1, S_{22}^{-1/2} B_2), \qquad K_{12} = \operatorname{tr} B_1' S_{11}^{-1/2} S_{12} S_{22}^{-1/2} B_2 \Lambda, \tag{4.11}$$

$$K_{ii} = -\tfrac{1}{2} \operatorname{tr} B_i S_{ii}^{-1/2} W_i' H_i W_i S_{ii}^{-1/2} B_i \Delta_i \qquad (i = 1, 2).$$

Hence $N_0 = c\, (\prod_{i=1}^{2} |S_{ii}|^{-n_i/2})$ in the ratio. We now expand $\exp[K]$ using

$$\exp[K_{ii} + f_{ii}^*] = 1 + K_{ii} + f_{ii}^* + o((K_{ii} + f_{ii}^*)^2) \qquad (i = 1, 2),$$

$$\exp[K_{12} + f_{12}^*] = 1 + K_{12} + f_{12}^* + \tfrac{1}{2}(K_{12} + f_{12}^*)^2 + o((K_{12} + f_{12}^*)^3),$$

and evaluate the integral of

$$\begin{aligned} &(1 + K_{11} + f_{11}^* + o_1)(1 + K_{22} + f_{22}^* + o_2)(1 + K_{12} + f_{12}^* + \tfrac{1}{2}(K_{12} + f_{12}^*)^2 + o_3) \\ &= 1 + \sum_{i=1}^{2} (K_{ii} + f_{ii}^*) + K_{12} + f_{12}^* + \tfrac{1}{2}(K_{12} + f_{12}^*)^2 + o_4 \end{aligned}$$

with respect to the measure $\exp[-\frac{1}{2}\sum_{i=1}^{2} \operatorname{tr} B_i' B_i]\, \chi(g)\, \nu(dg)/N_0$, where $o_i = o((K_{ii} + f_{ii})^2)$ for $i = 1, 2$, and $o_3 = o((K_{12} + f_{12})^3)$. To do so, decompose dB_i as $dB_i = d\mu_i \times dT_i$, where μ_i is the invariant probability measure on $\mathscr{O}(p_i)$ and dT_i is the Lebesgue measure on $G_T^+(p_i)$, the set of $p_i \times p_i$ lower triangular matrices with positive diagonal elements. Then writing $B_i = F_i T_i$ with $(F_i, T_i) \in \mathscr{O}(p_i) \times G_T^+(p_i)$ and $m_i = (n_i - p_i)/2$,

$$N_\Lambda/N_0 = |A|^{-1/2} \int_{G_T^+(p_1)\times G_T^+(p_2)} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{2} \operatorname{tr} T_i' T_i\right] \prod_{i=1}^{2} |T_i' T_i|^{m_i}\, dT_i$$
$$\times \int_{\mathcal{O}(p_1)\times\mathcal{O}(p_2)} \left[1 + \sum_{i=1}^{2} (K_{ii} + f_{ii}^*) + K_{12} + f_{12}^* \right.$$
$$\left. + \tfrac{1}{2}(K_{12} + f_{12}^*)^2 + o_4 \right] \mu_1(dF_1)\,\mu_2(dF_2)/M_0. \tag{4.12}$$

It is seen (e.g., see Kariya [3]) that

$$\int_{\mathcal{O}(p_i)} K_{ii}\mu_i(dF_i) = -\tfrac{1}{2} \operatorname{tr} S_{ii}^{-1} W_i' H_i W_i \operatorname{tr} T_i' T_i \Delta_i/p_i \qquad (i = 1, 2), \tag{4.13}$$

$$\int_{\mathcal{O}(p_1)\times\mathcal{O}(p_2)} K_{12}^2 \mu_1(dF_1)\,\mu_2(dF_2) = \operatorname{tr} S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} \operatorname{tr} T_2' T_2 \Delta' T_1' T_1 \Delta/p_1 p_2. \tag{4.14}$$

Further, since K_{12} is an odd function of F_1 (or F_2), the integral of K_{12} over $\mathcal{O}(p_1)$ (or $\mathcal{O}(p_2)$) is zero.

Next, we integrate (4.13) and (4.14) with respect to T_i's. From (4.12), it is easy to see that $T_i' T_i$'s follow $\mathscr{W}_{p_i}(I, n_i)$ independently. Hence the integrals of (4.13) and (4.14) over $G_T^+(p_1) \times G_T^+(p_2)$ are, respectively, equal to

$$-(n_i/2p_i) \operatorname{tr} S_{ii}^{-1} W_i' H_i W_i \operatorname{tr} \Delta_i \quad \text{and} \quad (n_1 n_2/p_1 p_2) \operatorname{tr} S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} \operatorname{tr} \Delta_1.$$

Consequently, using $|A|^{-1/2} = 1 + \frac{1}{2} \operatorname{tr} H_1 \operatorname{tr} \Delta_1 + o(\operatorname{tr} \Delta_1)$ uniformly, (4.12) becomes

$$N_\Lambda/N_0 = 1 + \tfrac{1}{2} J\tau + \Psi, \tag{4.15}$$

where J is given by (3.9), $\operatorname{tr} H_1 = \operatorname{tr} H_2 = \operatorname{tr} Q_1 Q_2$, $\tau = \operatorname{tr} \Delta_1 = \operatorname{tr} \Delta_2$, and Ψ is the sum of the integrals of all remaining terms.

Third, we evaluate the remainder terms. Note that $(\operatorname{tr} C^{ii})(I \otimes I) - C^{ii}$ is nonnegative definite and that $[H_i^2 \otimes \Delta_i^2] C^{ii} = [H_i \otimes \Delta_i]\, C^{ii} [H_i \otimes \Delta_i]$. Hence from (4.9) and (4.11), with $B_i = F_i T_i$, it is shown via Schwarz's inequality that

$$|f_{ii}^*| \leqslant k_{ii} \operatorname{tr} T_i' T_i \Delta_i^2 \leqslant k_{ii} \operatorname{tr} \Delta_i^2 \operatorname{tr} T_i' T_i, \tag{4.16}$$

$$|f_{12}^*|^2 \leqslant k_{12} \operatorname{tr} T_1' T_1 \Delta_1^2 \operatorname{tr} T_2' T_2 \leqslant k_{12} (\operatorname{tr} \Delta_1)^2 \operatorname{tr} T_1' T_1 \operatorname{tr} T_2' T_2, \tag{4.17}$$

for $\tau = \operatorname{tr} \Delta_i = \operatorname{tr} \Delta\Delta' < \tau_0$ and some constants k_{ij}, where from $Z_i' Z_i = I_{n_i}$, $|\operatorname{tr} C^{ii}| < k_1$, and $|\operatorname{tr} S_{ii}^{-1/2} W_i' H_i^2 W_i S_{ii}^{-1/2}| < k_2$ for some k_1 and k_2 is used.

Therefore, since from (4.11), $|K_{12}|^2 \leqslant k_{12}' \operatorname{tr} \Delta_1 \operatorname{tr} T_1' T_1 \operatorname{tr} T_2' T_2$,

$$|K_{12} f_{12}^*| \leqslant (k_{12} k_{12}')^{1/2} (\operatorname{tr} \Delta_1)^{3/2} \operatorname{tr} T_1' T_1 \operatorname{tr} T_2' T_2$$

for $\tau < \tau_0$. Using (4.16) and (4.17) and noting that $T_i' T_i \sim \mathscr{W}_{p_i}(I, n_i)$ independently, it is shown that all the remaining terms are bounded by $o(\tau)$ uniformly when they are integrated over $G_T^+(p_1) \times G_T^+(p_2)$. The details are similar to Kariya [3].

Thus, in (4.15), $\Psi = o(\tau)$ uniformly and hence for any invariant test ϕ of level α, the power function is evaluated

$$E_\Lambda[\phi] = \int \phi \, dP_\Lambda^T = \int \phi(dP_\Lambda^T / dP_0^T) \, dP_0^T$$

$$= \alpha + \tfrac{1}{2}\tau E_0[\phi J] + o(\tau).$$

Here applying the generalized Neyman–Pearson lemma to maximize $E_0[\phi J]$ yields the result. This completes the proof of Theorem 3.1.

5. Null Distributions of Test Statistics for Independence

In this section, we derive the null distributions of the following statistics useful in testing for the independence of two multivariate regression equations with different design matrices,

$$T_1 = -n_0 \log |I - R|, \tag{5.1}$$

$$T_2 = n_0 \operatorname{tr} R, \tag{5.2}$$

$$T_3 = \frac{1}{n_0} \{n_1 n_2 \operatorname{tr} R - n_1 p_2 \operatorname{tr} S_{11}^{-1} Y_1' Q_1 Q_2 Q_1 Y_1 - n_2 p_1 \operatorname{tr} S_{22}^{-1} Y_2' Q_2 Q_1 Q_2 Y_2\}, \tag{5.3}$$

where $n_0 = n - r_0$, $n_i = n - r_i$,

$$R = S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}, \tag{5.4}$$

Q_i $(i = 1, 2)$ is given by (2.12), and r_0 is the rank of $[X_1, X_2]$. The critical region of the test associated with T_i is given by $T_i > c$. The test based upon T_3 is the LBI test derived in Section 3. All the three tests are invariant. So, without loss of generality, we assume that under the null hypothesis $\Sigma_{12} = 0$,

$$\Sigma = \begin{pmatrix} I_{p_1} & 0 \\ 0 & I_{p_2} \end{pmatrix}.$$

Let L_i: $n \times r_i$ be a matrix satisfying the relations $P_i = L_i L_i'$ and $L_i' L_i = I_{r_i}$ for $i = 1, 2$. Also, let Z_0: $n \times (n - r_0)$ be a matrix satisfying

$$Q_0 = Z_0 Z_0' \qquad \text{and} \qquad Z_0' Z_0 = I_{n_0}, \tag{5.5}$$

where $Q_0 = I_n - X(X'X)^{+1} X'$, and $X = [X_1, X_2]$, where A^+ denotes the Penrose inverse of A. Further, let $\bar{Q}_j$ be the projection matrices onto $\mathscr{L}(X) \cap \mathscr{L}(Q_j)$ $(j = 1, 2)$, where $\mathscr{L}(A)$ denotes the column space of the matrix A. In addition, let $\bar{Z}_j$ be a matrix satisfying

$$\bar{Q}_j = \bar{Z}_j \bar{Z}_j' \qquad \text{and} \qquad \bar{Z}_j' \bar{Z}_j = I_{r_0 - r_j}. \tag{5.6}$$

Then, the matrices defined by

$$Z_1 = [\bar{Z}_1, Z_0], \qquad Z_2 = [\bar{Z}_2, Z_0] \tag{5.7}$$

satisfy $Z_i' Z_i = I_{n_i}$ and $Z_i Z_i' = Q_i = I - P_i$; this is a special choice of Z_i defined in (3.4). Next, let

$$W_i = Z_i' Y_i = \begin{bmatrix} M_i \\ U_i \end{bmatrix}, \tag{5.8}$$

where M_i is of order $(r_0 - r_i) \times p_i$ and U_i is of order $(n - r_0) \times p_i$. When $\Sigma_{12} = 0$, the rows of W_i are independently and identically distributed (i.i.d.) as multivariate normal with mean vector $\mathbf{0}$ and covariance matrix I_{p_i} and W_1 and W_2 are distributed independently of each other. From (3.7) and (5.8), we obtain

$$S = G + B, \tag{5.9}$$

where

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = [U_1\ U_2]'\ [U_1\ U_2], \tag{5.10}$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} M_1' M_1 & M_1' K M_2 \\ M_2' K' M_1 & M_2' M_2 \end{pmatrix}, \tag{5.11}$$

and $K = \bar{Z}_1' \bar{Z}_2$. It is seen that G and B are independent of each other since (M_1, M_2) and (U_1, U_2) are independent.

The above results can be summarized in

LEMMA 5.1. *The matrix S defined by* (3.7) *can be expressed as $S = G + B$, where G and B are defined by* (5.10) *and* (5.11), *respectively. When $\Sigma_{12} = 0$, G is distributed as $W_{p_1+p_2}(I, n_0)$, and M_1 is distributed as $N(0, I_{r_0 - r_i} \otimes I_{p_i})$ $(i = 1, 2)$ where M_i is defined by* (5.8). *Also G and B are distributed independently of each other when $\Sigma_{12} = 0$.*

The matrices $Y_1' Q_1 Q_2 Q_1 Y_1$ and $Y_2' Q_2 Q_1 Q_2 Y_2$ can be expressed as

$$Y_1' Q_1 Q_2 Q_1 Y_1 = S_{11} + M_1'(KK' - I_{r_0 - r_1}) M_1, \tag{5.12}$$

$$Y_2' Q_2 Q_1 Q_2 Y_2 = S_{22} + M_2'(K'K - I_{r_0 - r_1}) M_2. \tag{5.13}$$

We now derive the asymptotic null distributions of T_1, T_2, and T_3 under the assumption that

$$K = \bar{Z}_1'\bar{Z}_2 = O(1) \qquad \text{as } n_0 \to \infty. \tag{5.14}$$

The above assumption is satisfied in many applications. We first give an expansion of $R = S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}$.

A. *Expansion of R*

Let

$$\begin{aligned} V &= \sqrt{n_0}\left(\frac{1}{n_0}G - I_{p_1+p_2}\right) \\ &= \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \end{aligned} \tag{5.15}$$

where V_{ij} is of order $p_i \times p_j$. Then

$$R = \frac{1}{n_0}\left[R^{(0)} + \frac{1}{\sqrt{n_0}}R^{(1)} + \frac{1}{n_0}R^{(2)} + \cdots\right], \tag{5.16}$$

where $R^{(0)} = V_{12}V_{21}$,

$$R^{(1)} = R_1^{(1)} + R_2^{(1)}, \tag{5.17}$$

$$R_1^{(1)} = -V_{12}V_{22}V_{21} - V_{12}V_{21}V_{11}, \qquad R_2^{(1)} = V_{12}B_{21} + B_{12}V_{21}, \tag{5.18}$$

$$R^{(2)} = R_1^{(2)} + R_2^{(2)}, \tag{5.19}$$

$$R_1^{(2)} = V_{12}V_{21}V_{11}^2 + V_{12}V_{22}^2V_{21} + V_{12}V_{22}V_{21}V_{11}, \tag{5.20}$$

$$\begin{aligned} R_2^{(2)} = &-V_{12}V_{21}B_{11} - V_{12}B_{21}V_{11} + B_{12}B_{21} - B_{12}V_{21}V_{11} - V_{12}V_{22}B_{21} \\ &- V_{12}B_{22}V_{21} - B_{12}V_{22}V_{21}. \end{aligned} \tag{5.21}$$

B. *Asymptotic Distribution of T_1*

The test statistic T_1 defined by (5.1) can be expanded as

$$T_1 = \operatorname{tr} R^{(0)} + \frac{1}{\sqrt{n_0}}\operatorname{tr} R^{(1)} + \frac{1}{n_0}\left\{\operatorname{tr} R^{(2)} + \frac{1}{2}\operatorname{tr} R^{(0)2}\right\} + \cdots, \tag{5.22}$$

by using (5.16). Hence the characteristic function of T_1 is evaluated as

$$\begin{aligned} C_1(t) = E\Bigg[e^{it\operatorname{tr}R^{(0)}}\Bigg\{1 &+ \frac{1}{\sqrt{n_0}}it\operatorname{tr} R^{(1)} \\ &+ \frac{1}{n_0}\left[it\operatorname{tr} R^{(2)} + \frac{it}{2}\operatorname{tr}(R^{(0)})^2 + \frac{(it)^2}{2}(\operatorname{tr} R^{(1)})^2\right]\Bigg\}\Bigg] + o(n_0^{-1}) \\ = C_1^*(t) &+ C_1^{**}(t) + o(n_0^{-1}), \end{aligned} \tag{5.23}$$

where

$$C_1^*(t) = E\left[e^{it\operatorname{tr}R^{(0)}}\left\{1 + \frac{it}{\sqrt{n_0}}\operatorname{tr}R_1^{(1)} + \frac{1}{n_0}\left[it\operatorname{tr}R_1^{(2)} + \frac{it}{2}\operatorname{tr}(R^{(0)})^2 + \frac{(it)^2}{2}(\operatorname{tr}R_1^{(1)})^2\right]\right\}\right] \tag{5.24}$$

$$C_1^{**}(t) = E\left[e^{it\operatorname{tr}R^{(0)}}\left\{\frac{it}{\sqrt{n_0}}\operatorname{tr}R_2^{(1)} + \frac{1}{n_0}\left[it\operatorname{tr}R_2^{(2)} + \frac{(it)^2}{2}(\operatorname{tr}R_2^{(1)})^2 + (it)^2(\operatorname{tr}R_1^{(1)})(\operatorname{tr}R_2^{(1)})\right]\right\}\right]. \tag{5.25}$$

Since the characteristic function of $-n_0 \log |I - G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}|$ is $C_1^*(t) + o(n_0^{-1})$, we know (e.g., see Box [1]) that

$$C_1^*(t) = (1-2it)^{-f/2}\left[1 - \frac{1}{4n_0}f(p_1+p_2+1)\{1-(1-2it)^{-1}\}\right] + o(n_0^{-1}), \tag{5.26}$$

where $f = p_1 p_2$. We will now evaluate $C_1^{**}(t)$.

From Lemma 5.1 taking expectations with respect to M_i's only yields

$$\begin{aligned} E_M[\operatorname{tr}R_2^{(1)}] &= 2E[\operatorname{tr}V_{12}B_{21}] = 0,\\ E_M[(\operatorname{tr}R_1^{(1)})(\operatorname{tr}R_2^{(1)})] &= 0,\\ E[\operatorname{tr}R_2^{(2)}] &= p_1 p_2 \operatorname{tr}KK' + (r_1 + r_2 - 2r_0)\operatorname{tr}V_{12}V_{21},\\ E[(\operatorname{tr}R_2^{(1)})^2] &= 4\operatorname{tr}KK'\operatorname{tr}V_{12}V_{21}. \end{aligned} \tag{5.27}$$

So, from (5.25), we obtain

$$C_1^{**}(t) = \frac{1}{n_0}E[e^{it\operatorname{tr}V_{12}V_{21}}\{(it)p_1p_2\operatorname{tr}KK' + [it(r_1+r_2-2r_0) + 2(it)^2\operatorname{tr}KK']\operatorname{tr}V_{12}V_{21}\}]. \tag{5.28}$$

Here we note the limiting distribution of $V = (v_{ij})$ in (5.15) is the distribution of $\bar{V} = (\bar{v}_{ij})$, where $\bar{v}_{ii} \sim N(0, 2)$, $\bar{v}_{ij} \sim N(0, 1)$ $(i \neq j)$, and $\bar{v}_{ij}$ $(i \leqslant j)$'s are all independent, and that the density f of V can be expressed as

$$f(V) = f_0(V) + \frac{1}{\sqrt{n_0}}f_1(V) + \frac{1}{n_0}f_2(V) + \cdots, \tag{5.29}$$

where $f_0(\bar{V})$ is the pdf of $\bar{V}$, i.e.,

$$f_0(V)=\left\{\prod_{i=1}^{p_1+p_2}\frac{1}{\sqrt{4\pi}}\{\exp(-v_{ii}^2/4)\}\right\}\left\{\prod_{i-1}^{p_1+p_2}\frac{1}{\sqrt{2\pi}}\{\exp(-v_{ij}^2/2)\}\right\}.$$

Hence

$$\begin{aligned}
&E[e^{it\operatorname{tr}V_{12}V_{21}}\operatorname{tr}V_{12}V_{21}]\\
&\quad=\int e^{it\operatorname{tr}V_{12}V_{21}}(\operatorname{tr}V_{12}V_{21})f_0(V)\,dV+O(n_0^{-1/2}) \qquad (5.30)\\
&\quad=E[e^{it\operatorname{tr}XX'}(\operatorname{tr}XX')]+O(n_0^{-1/2})\\
&\quad=(1-2it)^{-f/2}(1-2it)^{-1}f+O(n_0^{-1/2}),
\end{aligned}$$

where $X\colon p_1\times p_2\sim N(0,I_{p_1}\otimes I_{p_2})$. So from (5.28) and (5.30), we obtain

$$\begin{aligned}
C_1^{**}(t)=\frac{p_1p_2}{2n_0}&(1-2it)^{-f/2}\,[r_1+r_2-2r_0+\operatorname{tr}KK']\\
&\times[(1-2it)^{-1}-1]+o(n_0^{-1}). \qquad (5.31)
\end{aligned}$$

Using (5.23), (5.26), and (5.31), we obtain

$$C_1(t)=(1-2it)^{-f/2}\left[1-\left(\frac{f}{2n_0}a\right)\{1-(1-2it)^{-1}\}\right]+o(n_0^{-1}), \quad (5.32)$$

$$a=\tfrac{1}{2}(p_1+p_2+1)+r_1+r_2-2r_0+\operatorname{tr}KK'. \quad (5.33)$$

Now inverting the right side of (5.32) yields the following expression for the asymptotic distribution of T_1,

$$P(T_1\leqslant x)=G_f(x)-\frac{f}{2n_0}a[G_f(x)-G_{f+2}(x)]+o(n_0^{-1}), \quad (5.34)$$

where $G_\beta(x)$ denotes the distribution function of χ^2-distribution with d.f. β and $f=p_1p_2$. But looking into (5.32) and modifying T_1 as

$$\tilde{T}_1=-\tilde{n}_0\log|I-S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}| \quad (5.35)$$

with $\tilde{n}_0=n_0-a$, the distribution of $\tilde{T}_1$ can be written as

$$P(\tilde{T}_1\leqslant x)=G_f(x)+o(n_0^{-1}). \quad (5.36)$$

This is obtained simply by changing t into $t\tilde{n}_0/n_0$ in (5.32) and inverting it. In other words, if we test the hypothesis $\Sigma_{12}=0$ based on $\tilde{T}_1$ rather than on T_1, we can use χ^2-approximation with d.f. $f=p_1p_2$ up to $O(n_0^{-1})$.

C. *Asymptotic Distribution of* T_2

Using (5.16), the statistic T_2 defined by (5.2) can be expressed as

$$T_2 = \operatorname{tr} R^{(0)} + \frac{1}{\sqrt{n_0}} \operatorname{tr} R^{(1)} + \frac{1}{n_0} \operatorname{tr} R^{(2)} + \cdots . \tag{5.37}$$

Hence the characteristic function of T_2 is

$$\begin{aligned} C_2(t) &= E\left[e^{it \operatorname{tr} R^{(0)}} \left\{ 1 + \frac{1}{\sqrt{n_0}} it \operatorname{tr} R^{(1)} + \frac{1}{n_0} \right.\right. \\ &\qquad \left.\left. \times \left[it \operatorname{tr} R^{(2)} + \frac{(it)^2}{2} (\operatorname{tr} R^{(1)})^2 \right] \right\} \right] + o(n_0^{-1}) \\ &= C_2^*(t) + C_2^{**}(t) + o(n_0^{-1}), \end{aligned} \tag{5.38}$$

where

$$\begin{aligned} C_2^*(t) &= E\left[e^{it \operatorname{tr} R^{(0)}} \left\{ 1 + \frac{1}{\sqrt{n_0}} it \operatorname{tr} R_1^{(1)} \right.\right. \\ &\qquad \left.\left. + \frac{1}{n_0} \left[it \operatorname{tr} R_1^{(2)} + \frac{(it)^2}{2} (\operatorname{tr} R_1^{(1)})^2 \right] \right\} \right] \\ &= E[e^{itu_0 \operatorname{tr} G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}}] + o(n_0^{-1}) \\ &= (1 - 2it)^{-f/2} \left[1 - \frac{f}{4n_0} (p_1 + p_2 + 1) \right. \\ &\qquad \left. \times \{1 - 2(1 - 2it)^{-1} + (1 - 2it)^{-2}\} \right] + o(n_0^{-1}) \end{aligned} \tag{5.39}$$

(see Fujikoshi (1970)), and

$$\begin{aligned} C_2^{**}(t) = C_1^{**}(t) &= \frac{f}{2n_0} (1 - 2it)^{-f/2} [r_1 + r_2 - 2r_0 + \operatorname{tr} KK'] \\ &\quad \times [(1 - 2it)^{-1} - 1] + o(n_0^{-1}). \end{aligned} \tag{5.40}$$

Using (5.39) and (5.40) and inverting (5.38), we obtain

$$\begin{aligned} P[T_2 \leqslant x] &= G_f(x) - \frac{f}{4n_0} (p_1 + p_2 + 1) [G_f(x) - 2G_{f+2}(x) \\ &\quad + G_{f+4}(x)] - \frac{f}{2n_0} (r_1 + r_2 - 2r_0 + \operatorname{tr} KK') \\ &\quad \times [G_{f+2}(x) - G_f(x)] + o(n_0^{-1}). \end{aligned} \tag{5.41}$$

D. *Asymptotic Distribution of* T_3

Again from (5.16), we obtain the following expansion of T_3, where T_3 was defined by (5.3)

$$\frac{n_1 n_2}{n_0} \operatorname{tr} R = \operatorname{tr} R^{(0)} + \frac{1}{\sqrt{n_0}} \operatorname{tr} R^{(1)} + \frac{1}{n_0} \operatorname{tr} R^{(2)} + \frac{1}{n_0} (2r_0 - r_1 - r_2) \operatorname{tr} R^{(0)} + \cdots . \tag{5.42}$$

Also,

$$\begin{aligned} &p_2 n_1 \operatorname{tr} S_{11}^{-1} Y_1' Q_1 Q_2 Q_1 Y_1 / n_0 \\ &= p_2 \left[1 + \frac{1}{n_0} (r_0 - r_1) \right] \operatorname{tr}[I_{p_1} + \operatorname{tr} S_{11}^{-1} M_1'(KK' - I) M_1] \\ &= p_1 p_2 + \frac{1}{n_0} [p_1 p_2 (r_0 - r_1) + p_2 \operatorname{tr} M_1'(KK' - I) M_1] + \cdots . \end{aligned} \tag{5.43}$$

Similarly we obtain

$$\begin{aligned} &n_2 p_1 \operatorname{tr} S_{22}^{-1} Y_2' Q_2 Q_1 Q_2 Y_2 / n_0 \\ &= p_1 p_2 + \frac{1}{n_0} [p_1 p_2 (r_0 - r_2) + p_1 \operatorname{tr} M_2'(KK' - I) M_2] + \cdots . \end{aligned} \tag{5.44}$$

Hence T_3 is expressed as

$$\begin{aligned} T_3 = {} & \operatorname{tr} R^{(0)} + \frac{1}{\sqrt{n_0}} \operatorname{tr} R^{(1)} + \frac{1}{n_0} \operatorname{tr} R^{(2)} \\ & + \frac{1}{n_0} (2r_0 - r_1 - r_2) \operatorname{tr} R^{(0)} \\ & - p_1 p_2 - \frac{1}{n_0} [p_1 p_2 (r_0 - r_1) + p_2 \operatorname{tr} M_1'(KK' - I) M_1] \\ & - p_1 p_2 - \frac{1}{n_0} [p_1 p_2 (r_0 - r_2) + p_1 \operatorname{tr} M_2'(K'K - I) M_2] + \cdots . \end{aligned} \tag{5.45}$$

Based on this expression, we modify T_3 into $\hat{T}_3$ defined by

$$\begin{aligned} \hat{T}_3 &= T_3 + p_1 p_2 \left[2 + \frac{1}{n_0} (2r_0 - r_1 - r_2) \right] \\ &= \operatorname{tr} R^{(0)} + \frac{1}{\sqrt{n_0}} \operatorname{tr} R^{(1)} + \frac{1}{n_0} \operatorname{tr} R^{(2)} \\ &\quad + \frac{1}{n_0} (2r_0 - r_1 - r_2) \operatorname{tr} R^{(0)} \\ &\quad - \frac{1}{n_0} [p_2 \operatorname{tr} M_1'(KK' - I) M_1 + p_1 \operatorname{tr} M_2'(K'K - I) M_2] + \cdots . \end{aligned} \tag{5.46}$$

Then the characteristic function of $\hat{T}_3$ is given by

$$\hat{C}_3(t) = C_2(t) + E\left[e^{it\operatorname{tr}R^{(0)}}\frac{(it)}{n_0}\{(2r_0 - r_1 - r_2)\operatorname{tr}R^{(0)}\right.$$

$$\left. - p_2 \operatorname{tr} M_1'(KK' - I)M_1 - p_1 \operatorname{tr} M_2'(K'K - I)M_2\}\right]$$

$$= (1-2it)^{-f/2}\left[1 - \frac{f}{4n_0}(p_1 + p_2 + 1)\{1 - 2(1-2it)^{-1}\right.$$

$$+ (1-2it)^2\} - \frac{f}{2n_0}\operatorname{tr}KK'\{1-(1-2it)^{-1}\}$$

$$\left. + (it)\frac{f}{n_0}(2r_0 - r_1 - r_2 - 2\operatorname{tr}KK')\right] + o(n_0^{-1}). \tag{5.47}$$

Looking into this expression, we modify $\hat{T}_3$ into

$$\tilde{T}_3 = \hat{T}_3 - \frac{f}{n_0}(2r_0 - r_1 - r_2 - 2\operatorname{tr}KK')$$

$$= T_3 + 2f\left[1 + \frac{1}{n_0}\operatorname{tr}KK'\right], \tag{5.48}$$

Then the characteristic function of $\tilde{T}_3$ is

$$(1-2it)^{-f/2}\left[1 - \frac{f}{4n_0}(p_1+p_2+1)\{1-2(1-2it)^{-1} + (1-2it)^{-2}\}\right.$$

$$\left. - \frac{f}{2n_0}\operatorname{tr}KK'\{1-(1-2it)^{-1}\}\right] + o(n_0^{-1}). \tag{5.49}$$

Therefore the asymptotic expansion of the null distribution of $\tilde{T}_3$ is given by

$$P(\tilde{T}_3 \leqslant x) = G_f(x) - \frac{f}{4n_0}(p_1+p_2+1)\{G_f(x) - 2G_{f+2}(x) + G_{f+4}(x)\}$$

$$- \frac{p_1 p_2}{2n_0}\operatorname{tr}KK'\{G_f(x) - G_{f+2}(x)\} + o(n_0^{-1}). \tag{5.50}$$

THEOREM 5.1. (1) *The null distribution of the* LRT-*like test based on* $\tilde{T}_1$ *in* (5.35) *is approximated up to* $O(n_0^{-1})$ *by* χ^2 *distribution with degrees of freedom* $f = p_1 p_2$.

(2) *The null distribution of the trace test based on* T_2 *in* (5.2) *is approximated up to* $O(n_0^{-1})$ *by* (5.41).

(3) *The null distribution of the LBI test based on* $\tilde{T}_3$ *in* (5.48) *is approximated up to* $O(n_0^{-1})$ *by* (5.50).

6. Nonnull Distributions of Test Statistics for Independence

In this section, we derive the nonnull distributions of the test statistics given in (5.1), (5.2), and (5.3). Let W_i, $Z_i\bar{Z}_i$, Z_0, M_i, and U_i be defined as in the preceding section. We know that G and B are distributed independently of each other. Also,

$$\begin{bmatrix} \mathbf{m}_1^{(1)} \\ \vdots \\ \mathbf{m}_1^{(r_0-r_1)} \\ \mathbf{m}_2^{(1)} \\ \vdots \\ \mathbf{m}_2^{(r_0-r_2)} \end{bmatrix} \sim N\left(\mathbf{0}, \begin{bmatrix} I\otimes\Sigma_{11} & K\otimes\Sigma_{12} \\ K'\otimes\Sigma_{21} & I\otimes\Sigma_{22} \end{bmatrix}\right),$$

where $\mathbf{m}_i^{(j)}$ denotes the jth column of M_i'. Without loss of generality, we assume that

$$\Sigma = \begin{pmatrix} I & \Lambda \\ \Lambda' & I \end{pmatrix}, \qquad (p_1 \geqslant p_2)$$

where $\rho_1^2 \geqslant \cdots \geqslant \rho_{p_2}^2$ are the nonzero characteristic roots of $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}$ and (Λ was derived in (3.6). We now derive the asymptotic distributions of T_i ($i = 1, 2, 3$) under the local alternatives

$$\Lambda = \frac{1}{\sqrt{n_0}}\Xi = \frac{1}{\sqrt{n_0}}\begin{bmatrix} D_\xi \\ 0 \end{bmatrix},$$

where $D_\xi = \mathrm{diag}(\xi_1, ..., \xi_{p_2})$ and ξ_i's are fixed. We need the following in the sequel. Let

$$\frac{1}{n_0}G = \Sigma + \frac{1}{\sqrt{n_0}}V = I + \frac{1}{\sqrt{n_0}}\begin{bmatrix} 0 & \Xi \\ \Xi' & 0 \end{bmatrix} + \frac{1}{\sqrt{n_0}}\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}. \quad (6.1)$$

The pdf of $V = \{v_{ij};\ i \leqslant j\}$ can be expanded as

$$\tilde{f}(\mathbf{V}) = f_0(V) + \frac{1}{\sqrt{n_0}} \tilde{f}_1(V) + \frac{1}{n_0} \tilde{f}_2(V) + \cdots, \tag{6.2}$$

where $f_0(V)$ is the same as in (5.29). Next, let

$$\tilde{V}_{12} = V_{12} + \Xi. \tag{6.3}$$

Then, we can express $R = S_{12} S_{22}^{-1} S_{21} S_{11}^{-1}$ as

$$R = \frac{1}{n_0} \left(\tilde{R}^{(0)} + \frac{1}{\sqrt{n_0}} \tilde{R}^{(1)} + \frac{1}{n_0} \tilde{R}^{(2)} + \cdots \right), \tag{6.4}$$

where $\tilde{R}^{(i)} = \tilde{R}_1^{(i)} + \tilde{R}_2^{(i)}$ and $\tilde{R}_j^{(i)}$ is defined from $R_j^{(i)}$ by replacing V_{12} by $\tilde{V}_{12}$.

A. *Asymptotic Distribution of* T_1

The characteristic function of T_1 is

$$C_1(t) = \tilde{C}_1^*(t) + \tilde{C}_1^{**}(t) + o(n_0^{-1}), \tag{6.5}$$

where $\tilde{C}_1^*(t)$ and $\tilde{C}_1^{**}(t)$ are defined from the $C_1^*(t)$ and $C_1^{**}(t)$ in (5.24) and (5.25) by replacing $R_j^{(i)}$ by $\tilde{R}_j^{(i)}$. But

$$\begin{aligned} \tilde{C}_1^*(t) &= \text{the ch. f. of} - n_0 \log |I - G_{12} G_{22}^{-1} G_{21} G_{11}^{-1}| + o(n_0^{-1}) \qquad (6.6) \\ &= (1 - 2it)^{-f/2} e^{(it/(1-2it))\xi^2} \left[1 + \frac{1}{4n_0} \sum_{j=0}^{3} a_j (1 - 2it)^{-1} + o(n_0^{-1}) \right] \end{aligned}$$

(e.g., see Sugiura [11]),

$$\begin{aligned} f &= p_1 p_2, \qquad \xi^2 = \operatorname{tr} \Xi \Xi', \\ a_0 &= -f(p_1 + p_2 + 1) - \operatorname{tr}(\Xi \Xi')^2, \\ a_1 &= f(p_1 + p_2 + 1), \qquad a_2 = 2 \operatorname{tr} (\Xi \Xi')^2, \\ a_3 &= -\operatorname{tr}(\Xi \Xi')^2. \end{aligned}$$

We can show that

$$E_M[\operatorname{tr} \tilde{R}_2^{(1)}] = \frac{2}{\sqrt{n}} (\operatorname{tr} KK') \operatorname{tr} \tilde{V}_{21} \Xi, \tag{6.7}$$

$$E_M[\operatorname{tr} \tilde{R}_2^{(2)}] = (r_1 + r_2 - 2r_0) \operatorname{tr} \tilde{V}_{12} \tilde{V}_{21} + p_1 p_2 \operatorname{tr} KK' + O\left(\frac{1}{\sqrt{n_0}}\right), \tag{6.8}$$

$$E_M[(\operatorname{tr} \tilde{R}_2^{(1)})^2] = 4(\operatorname{tr} KK') \operatorname{tr} \tilde{V}_{12} \tilde{V}_{21} + O\left(\frac{1}{\sqrt{n_0}}\right), \tag{6.9}$$

$$E_M[(\operatorname{tr} \tilde{R}_1^{(1)}) \operatorname{tr} \tilde{R}_2^{(1)}] = O\left(\frac{1}{\sqrt{n_0}}\right). \tag{6.10}$$

So

$$\begin{aligned} \tilde{C}_1^{**}(t) &= \frac{1}{n_0} E[e^{it \operatorname{tr} \tilde{V}_{12}\tilde{V}_{21}}\{2(it)(\operatorname{tr} KK') \operatorname{tr} \tilde{V}_{21} \Xi \\ &\quad + (it) p_1 p_2 (\operatorname{tr} KK') \\ &\quad + [(it)(r_1 + r_2 - 2r_0) + 2(it)^2 (\operatorname{tr} KK')] \\ &\quad \times \operatorname{tr} \tilde{V}_{12} \tilde{V}_{21}\}] + o(n_0^{-1}) & (6.11) \\ &= (1 - 2it)^{-f/2} \exp(it\xi^2/(1 - 2it)) \\ &\quad \times \left[\frac{1}{2n} \sum_{j=0}^{2} d_j (1 - 2it)^{-j} + o(n^{-1}), \right. & (6.12) \end{aligned}$$

where

$$\begin{aligned} d_0 &= -f(r_1 + r_2 - 2r_0 + \operatorname{tr} KK') - (\operatorname{tr} KK') \operatorname{tr} \Xi\Xi' \\ d_1 &= f(r_1 + r_2 - 2r_0 + \operatorname{tr} KK') - (r_1 + r_2 - 2r_0) \operatorname{tr} \Xi\Xi' \\ d_2 &= (\mathrm{r}_1 + r_2 - 2r_0 + \operatorname{tr} KK') \operatorname{tr} \Xi\Xi' \\ \xi^2 &= \operatorname{tr} \Xi\Xi', \\ &\quad G_f(x : \xi^2) = P[\chi_f^2(\xi^2) \leqslant x]. \end{aligned}$$

Using (6.5), (6.6), and (6.11), we obtain

$$\begin{aligned} P[T_1 \leqslant x] &= G_f(x : \xi^2) \\ &\quad + \frac{1}{4n_0} \sum_{j=0}^{3} a_j G_{f+2j}(x : \xi^2) & (6.13) \\ &\quad + \frac{1}{2n_0} \sum_{j=0}^{2} d_j G_{f+2j}(x : \xi^2) + o(n_0^{-1}). \end{aligned}$$

When $\Xi = 0$, the formula (6.13) coincides with (5.34).

B. *Asymptotic Distribution of* T_2

The characteristic function of T_2 is given by

$$C_2(t) = \tilde{C}_2^*(t) + \tilde{C}_2^{**}(t) + o(n_0^{-1}), \tag{6.14}$$

where $\tilde{C}_2^*(t)$ and $\tilde{C}_2^{**}$ are defined from the $C_2^*(t)$ and $C_2^{**}(t)$ in (5.39) and (5.40) by replacing $R_j^{(i)}$ by $\tilde{R}_j^{(i)}$. We can see that

$$\begin{aligned} C_2^*(t) &= E[e^{itn\operatorname{tr} G_{12}G_{22}^{-1}G_{21}G_{11}^{-1}}] + o(n_0^{-1}) \\ &= (1-2it)^{-f/2}\, e^{(it/(1-2it))\xi^2} \\ &\quad \times \left[1 + \frac{1}{4n_0}\sum_{j=0}^{4} b_j(1-2it)^{-j} + o(n_0^{-1})\right] \end{aligned} \tag{6.15}$$

(Fujikoshi [2]) and the coefficients b_j are given by

$$\begin{aligned} b_0 &= -f(p_1+p_2+1) - \operatorname{tr}(\Xi\Xi')^2, \\ b_1 &= 2f(p_1+p_2+1), \\ b_2 &= -f(p_1+p_2+1) + 2(p_1+p_2+1)\operatorname{tr}\Xi\Xi' + 2\operatorname{tr}(\Xi\Xi')^2, \\ b_3 &= -2(p_1+p_2+1)\operatorname{tr}\Xi\Xi', \\ b_4 &= -\operatorname{tr}(\Xi\Xi')^2. \end{aligned}$$

Also,

$$\tilde{C}_2^{**}(t) = \tilde{C}_1^{**}(t).$$

So

$$\begin{aligned} P[T_2 \leqslant x] &= G_f(x\colon \xi^2) \\ &\quad + \frac{1}{4n_0}\sum_{j=0}^{4} b_j G_{f+2j}(x\colon \xi^2) \\ &\quad + \frac{1}{2n_0}\sum_{j=0}^{2} d_j G_{f+2j}(x\colon \xi^2) + o(n_0^{-1}). \end{aligned} \tag{6.16}$$

When $\Xi = 0$, the formula (6.16) coincides with (5.41).

C. *Expansion of* T_3

We consider the distribution of

$$\begin{aligned} \tilde{T}_3 &= T_3 + 2p_1p_2\left(1 + \frac{1}{n_0}\operatorname{tr} KK'\right) \\ &= \operatorname{tr}\tilde{R}^{(0)} + \frac{1}{\sqrt{n_0}}\operatorname{tr}\tilde{R}^{(1)} \\ &\quad + \frac{1}{n_0}\{\operatorname{tr}\tilde{R}^{(2)} + (2r_0 - r_1 - r_2)\operatorname{tr}\tilde{R}^{(0)} \\ &\quad - p_1p_2(2r_0 - r_1 - r_2 - 2\operatorname{tr} KK') \\ &\quad - p_2\operatorname{tr} M_1'(KK'-I)M_1 - p_1\operatorname{tr} M_2'(K'K-I)M_2\} \end{aligned} \tag{6.17}$$

instead of T_3. The characteristic function of $\tilde{T}_3$ can be written as

$$\tilde{C}_3(t) = E[e^{it\tilde{T}_3}] = \tilde{C}_3^*(t) + \tilde{C}_3^{**}(t) + o(n^{-1}), \tag{6.18}$$

where

$$\tilde{C}_3^*(t) = \tilde{C}_2^*(t) + \tilde{C}_1^{**}(t) \tag{6.19}$$

$$\begin{aligned} \tilde{C}_3^{**}(t) = \frac{1}{n_0} E[e^{it \operatorname{tr} \tilde{R}^{(0)}} \{ &(it)(2r_0 - r_1 - r_2) \operatorname{tr} \tilde{R}^{(0)} \\ &- (it) p_1 p_2 (2r_0 - r_1 - r_2 - 2 \operatorname{tr} KK') \\ &- (it) p_2 \operatorname{tr} M_1'(KK' - I) M_1 \\ &- (it) p_1 \operatorname{tr} M_2'(K'K - I) \}] + o(n_0^{-1}). \end{aligned} \tag{6.20}$$

After some computations, we obtain

$$\begin{aligned} \tilde{C}_3^{**}(t) &= \frac{1}{n_0} E[e^{it \operatorname{tr} R^{(0)}} (it)(2r_0 - r_1 - r_2) \operatorname{tr} R^{(0)}] \\ &= (1 - 2it)^{-f/2} e^{(it/(1-2it))\xi^2} \frac{1}{2n} (2r_0 - r_1 - r_2) \\ &\quad \times [\{(1 - 2it)^{-1} - 1\} f \\ &\quad + \{(1 - 2it)^{-2} - (1 - 2it)^{-1}\} \operatorname{tr} \Xi\Xi'] + o(n_0^{-1}). \end{aligned} \tag{6.21}$$

Using (6.18), (6.15), (6.12), (6.21), and inverting the characteristic function, we obtain the distribution of $\tilde{T}_3$.

$$\begin{aligned} P[\tilde{T}_3 \leqslant x] = G_f(x\colon \xi^2) &+ \frac{1}{4n_0} \sum_{j=0}^{4} b_j G_{f+2j}(x\colon \xi^2) \\ &+ \frac{1}{2n_0} \sum_{j=0}^{2} \tilde{d}_j G_{f+2j}(x\colon \xi^2) + o(n_0^{-1}), \end{aligned} \tag{6.22}$$

where

$$\begin{aligned} \tilde{d}_0 &= -(f + \operatorname{tr} \Xi\Xi') \operatorname{tr} KK', \\ \tilde{d}_1 &= f \operatorname{tr} KK', \\ \tilde{d}_2 &= \operatorname{tr} KK' \operatorname{tr} \Xi\Xi'. \end{aligned}$$

When $\Xi = 0$, the formula (6.22) coincides with (5.50).

7. Some Alternative Tests for Independence

In Sections 3–6, we discussed some test procedures and the distribution theory connected with these test procedures. These statistics are certain functions of the eigenvalues of the canonical correlation matrix $R = S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}$. We can use some other functions of the eigenvalues of R like the largest root, etc., as test statistics. We now discuss the problem of testing $\Sigma_{12} = 0$ against 1-sided alternatives.

Let H_j: $\mathbf{a}_j' \Sigma_{12} \mathbf{b}_j = 0$ and A_j: $\mathbf{a}_j' \Sigma_{12} \mathbf{b}_j > 0$ for $j = 1, 2,..., q$, where $\mathbf{a}_j'$ and $\mathbf{b}_j$ are known. Also, let

$$t_j = \frac{\mathbf{a}_j' S_{12} \mathbf{b}_j}{\sqrt{\mathbf{a}_j' S_{11} \mathbf{a}_j \mathbf{b}_j' S_{22} \mathbf{b}_j}}.$$

Then, we accept or reject H_j against A_j according as $t_j \lesseqgtr a$, where

$$P[t_j \leqslant a; j = 1, 2,..., q \mid \cap H_j] = (1 - \alpha). \tag{7.1}$$

We can similarly propose tests for $H_1,..., H_q$ against 2-sided alternatives or 1-sided alternatives of the form $\mathbf{a}_j' \Sigma_{12} \mathbf{b}_j < 0$. Similarly, some hypotheses can be tested against alternatives of the form $\mathbf{a}_j' \Sigma_{12} \mathbf{b}_j > 0$ and other hypotheses against alternatives of the form $\mathbf{a}_j' \Sigma_{12} \mathbf{b}_j \leqslant 0$. We can consider $\Sigma_{12} = 0$ against a combination of 1-sided and 2-sided alternatives also. The probability integral on the left side of (7.1) involves nuisance parameters. We can use Bonferroni's inequality to obtain a lower bound on the left side of (7.1) and this would require knowledge of the marginal distributions only. But the distribution of t_j^2 is of the same form (except for trivial modification) as the distribution of $\operatorname{tr} R$ when $p_1 = p_2 = 1$. Also, we can obtain the distribution function of t_j from the distribution function of t_j^2 since t_j is symmetric.

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